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Explaining the physics of tsunamis to undergraduate and non-physics students

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Abstract

After the recent devastating tsunami in southern Asia, I tried to find a simple way to present the physics of this phenomenon to students—in particular, the origin of the dispersion relation and the consequent wave properties. Being unable to locate a suitable source for a truly elementary approach, I developed the simple derivation presented here by extending to shallow-water waves a clever capillary wave model recently developed by Behroozi and Podolefsky (2001 *Eur. J. Phys.* **23** 225). The main properties of tsunami waves can thus be obtained with an elementary and straightforward procedure suitable for undergraduates in physics and other disciplines.

1. Introduction

The tsunami that devastated southern Asia on 26 December 2004 attracted the attention of a broad and diversified public to this extreme wave phenomenon. Many Internet sites can be found explaining the basic features that make it so destructive and distinguish it from other types of water waves: its causes, the extremely long wavelengths and the behaviour when it reaches a coastal region [2]. However, I could not find a source presenting in simple terms the physical origin of these characteristics and in particular of the dispersion relation that underlies the last one.

I propose here a very simple yet rigorous derivation, suitable for undergraduate physics students and also for students in other domains. This approach overcomes the difficulties of the two standard ways to present the subject. The first one [2] simply consists in introducing the link between the wave group velocity and the water depth without explaining its origin. This tactic is suitable for a very broad audience but unsatisfactory for college students since, for example, its starting hypotheses are not self-evident.

The second approach [3] is a complete derivation of the dispersion relation based on the (sometimes questioned [1]) irrotational character of the water motion and on mass conservation. Even with substantial simplifying assumption, the derivation is quite complex and involves non-intuitive notions such as the velocity potential. In several cases, derivations of this kind start by treating a very complicated model and then resort to approximations—simplifying the results only *a posteriori* [3].

The present simple approach was inspired by a brilliant solution recently elaborated by Behroozi and Podolefsky [1] for the problem of capillary gravity waves in deep water. With a similar strategy, my treatment leads to the dispersion relation for shallow-water waves and to the link between wave amplitude and energy—the essential elements to understand the destructive character of tsunamis.

It should be noted that a simple but correct way to comprehend tsunamis is important beyond mere scientific curiosity. A large portion of the casualties in south Asia was caused by trivial misunderstandings of the physical mechanism. For example, the lack of knowledge about large time intervals between different phases of the phenomenon prevented many people from taking simple life-saving precautions—such as leaving the beaches and getting to high ground after the first anomalous water retreat from the shore. Hopefully, a more widespread knowledge of the nature of tsunamis could mitigate their human impact.

2. The teaching strategy

The objective is to present to the students the following points establishing the physics background of tsunamis: (1) the typical wavelength magnitudes, (2) the differences between the wave behaviour far from a coast and in coastal areas, (3) the capacity to travel over very long distances.

2.1. Prerequisites

The students should have a general knowledge of elementary mechanics, a basic knowledge of wave properties (wavelength, frequency, period, the fact that they carry energy and momentum, the phase and group velocity) and some notion of fluid mechanics (such as the Bernoulli law and the notion of viscosity).

2.2. The model and its presentation

The introductory part of the presentation to students should stress the general characteristics of tsunamis and their difference with respect to the common waves observed near a beach. First the causes: tsunamis are initiated by earthquakes, landslides, volcanic eruptions, more rarely by meteorites and nuclear tests, and then sustained by gravity—whereas wind is an important factor in common beach waves [2]. This discussion should also argue that using the term ‘tidal waves’ for tsunamis is not appropriate since tides are not their origin.

The next step of the presentation should explain that tsunamis have very long wavelength λ and period. The λ-values typically range from kilometres to hundreds of kilometres, thus they are much longer than the hundreds of metres of common wind-generated waves near a beach. Since the ocean floor is nowhere deeper than 11 km (the Mariana trench), this implies that even in deep-sea areas tsunamis behave as shallow-water waves for which $\lambda/H > 1$ ($H =$ sea depth). When they reach a coastal area, their characteristics change but of course not the shallow-water nature.

The discussion should then move to the wave-related motion of water particles. On the basis of the students’ practical experience with real waves, one can lead them to understand that the local particle motion occurs along circular or elliptical paths. The students should

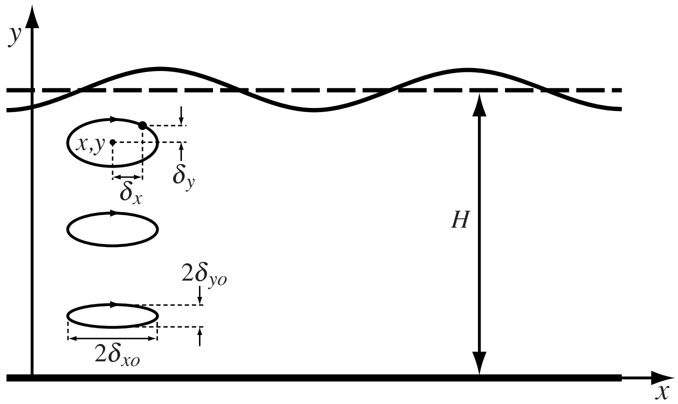

Figure 1. Water waves correspond to a local elliptical motion of the water particles. The figure defines the motion parameters used in the text: the average particle position (x, y), the shifts δ_x and δ_y with respect to the average position and the motion amplitudes δ_{xo} and δ_{yo}. H is the unperturbed water height.

have already seen in elementary mechanics that, for a constant average position (the centre of the trajectory), such a motion corresponds to an equation set as

$$\delta_x = \delta_{xo} \sin(\pm\omega t), \tag{1}$$

$$\delta_y = -\delta_{yo} \cos(\pm\omega t), \tag{2}$$

where δ_x and δ_y are the shifts with respect to the average position in a vertical x–y plane (see figure 1) and δ_{xo} and δ_{yo} are constants ($\delta_{xo} \neq \delta_{yo}$ for an elliptical motion). When the motion is associated with a wave propagating along the horizontal x-axis, equations (1) and (2) obviously become

$$\delta_x = \delta_{xo} \sin(kx \pm \omega t), \tag{3}$$

$$\delta_y = -\delta_{yo} \cos(kx \pm \omega t). \tag{4}$$

One should note at this point that equation (4) is affected by a problem: the vertical motion does not reduce to zero at the sea bottom ($y = 0$ m), as it should. To eliminate this problem, one must assume that δ_{yo} is not simply a constant but a decreasing function of the altitude y—see figure 1.

The simplest possible function is linear: $\delta_{yo} \propto$ constant $\times\ y$. One can specifically propose the linear form $\delta_{yo} = \delta_{xo} k y$ that has an additional advantage: being proportional to $k = 2\pi/\lambda$ it forces the vertical motion amplitude to remain limited. In fact, the maximum motion amplitude corresponds to the value of δ_{yo} for the sea top, $y = H$, thus it equals $\delta_{xo} kH = 2\pi\delta_{xo} H/\lambda$. Since in most cases $H/\lambda \ll 1$, this means that the vertical amplitude is smaller than the horizontal amplitude δ_{xo}, in agreement with the empirical observations[1].

In summary, the proposed local motion equations for the shallow water wave are

$$\delta_x = \delta_{xo} \sin(kx \pm \omega t), \tag{5}$$

$$\delta_y = -\delta_{xo} k y \cos(kx \pm \omega t). \tag{6}$$

Equation (6) gives in particular the water-surface wave when y is replaced by H: $\delta_y = -\delta_{xo} k H \cos(kx \pm \omega t)$.

[1] More formally, the $\delta_{yo} = \delta_{xo} k y$ factor guarantees that the velocity is consistent with the continuity equation.

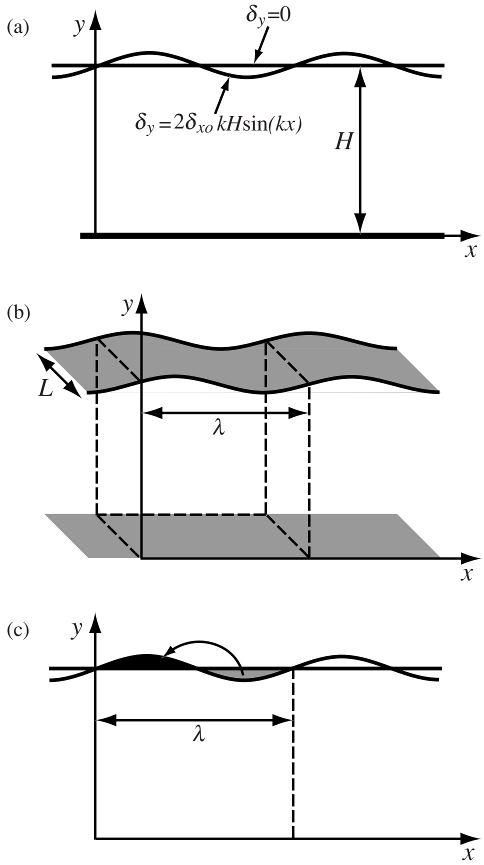

Figure 2. (a) Two important configurations of the stationary wave of equations (9) and (10): maximum deformation of the water surface (corresponding to zero velocity components in the entire water mass) and the flat-surface case (that maximizes the velocity magnitudes and the kinetic energy). (b) Definition of the water volume of width L and length λ for which the potential and kinetic energies are calculated. (c) The potential energy difference between the two configurations simply correspond to the energy required to lift water from the second half of the area to the first half.

Equations (5) and (6) correspond to the following velocity components:

$$v_x = \delta_{xo}(\pm\omega)\cos(kx \pm \omega t), \qquad (7)$$

$$v_y = \delta_{xo}ky(\pm\omega)\sin(kx \pm \omega t). \qquad (8)$$

The next objective is to find the dispersion relation, i.e., the link between ω and k. The procedure is simply based on the conservation of energy (assuming a non-compressible liquid with no viscosity) and can be greatly simplified—following the strategy of Behroozi and Podolefsky [1]—by treating a linear combination of the motion of equations (5) and (6) that produces a stationary wave:

$$\delta_x = \delta_{xo}[\sin(kx + \omega t) - \sin(kx - \omega t)] = 2\delta_{xo}\cos(kx)\sin(\omega t), \qquad (9)$$

$$\delta_y = -\delta_{xo}ky[\cos(kx + \omega t) - \cos(kx - \omega t)] = 2\delta_{xo}ky\sin(kx)\sin(\omega t). \qquad (10)$$

Consider now (figure 2(a)) two interesting configurations of this standing wave: first, for $\sin(\omega t) = 1$ and therefore $\cos(\omega t) = 0$, there is a maximum deformation of the water surface corresponding to $\delta_y = 2\delta_{xo}kH\sin(kx)$—whereas both velocity components are zero everywhere in the water mass. This means that there is only potential energy and not kinetic energy. In contrast, for $\sin(\omega t) = 0$ and therefore $\cos(\omega t) = 1$ the water surface is flat whereas the velocity magnitudes and the kinetic energy are maximized.

In the absence of dissipation due to viscosity, turbulence and other factors, the energy must be conserved. Therefore, the potential energy difference ΔU between the two configurations must correspond to the kinetic energy K in the second one.

This result can be quantitatively analysed in the case of a water volume of length λ (along the x-axis) and width L (figure 2(b)). The potential energy difference between the two configurations corresponds (figure 2(c)) to the energy necessary to raise water from the second half of the region to the first:

$$\Delta U = \int_0^{\lambda/2} \mathrm{d}x(\rho L g \delta_y)\delta_y = \int_0^{\lambda/2} \mathrm{d}x\left(\rho L g \delta_y^2\right) = 4\rho L g \delta_{xo}^2 k^2 H^2 \int_0^{\lambda/2} \sin^2(kx)\,\mathrm{d}x$$
$$= 2\rho L g \delta_{xo}^2 k H^2 \pi. \tag{11}$$

On the other hand, the kinetic energy for the second configuration can be obtained by integrating over the entire depth the kinetic energy of each mass element in the water volume. Equations (9) and (10) give the following two velocity components for $\cos(\omega t) = 1$:

$$v_x = 2\delta_{xo}\omega\cos(kx), \tag{12}$$

$$v_y = 2\delta_{xo}ky\omega\sin(kx). \tag{13}$$

Therefore

$$K = \frac{\rho L}{2}\int_0^H \mathrm{d}y \int_0^\lambda \mathrm{d}x\left(v_x^2 + v_y^2\right) = 2\rho L\omega^2\delta_{xo}^2\omega^2 \int_0^H \mathrm{d}y \int_0^\lambda \mathrm{d}x[\cos^2(kx) + k^2y^2\sin^2(kx)]$$
$$= \frac{2\rho L\omega^2\delta_{xo}^2\omega^2\pi}{k}\int_0^H \mathrm{d}y(1 + k^2y^2) = \frac{2\rho L\omega^2\delta_{xo}^2\omega^2\pi H}{k}\left(1 + \frac{k^2H^2}{3}\right),$$

and, since $kH = 2\pi H/\lambda \ll 1$

$$K \approx \frac{2\rho L\omega^2\delta_{xo}^2\omega^2\pi H}{k}. \tag{14}$$

By inserting equations (11) and (14) in the energy-conservation condition $\Delta U = K$, we then have

$$\omega^2 \approx (gH)k^2. \tag{15}$$

This dispersion relation contains all the elements to understand the basic properties of shallow-water waves and in particular of tsunamis.

2.3. Discussion

Equation (15) immediately leads to similar expressions for the magnitude of both the phase velocity and the group velocity:

$$|v_p| = \left|\frac{\omega}{k}\right| = \sqrt{gH}. \tag{16}$$

$$|v_g| = \left|\frac{\mathrm{d}\omega}{\mathrm{d}k}\right| = \sqrt{gH}. \tag{17}$$

As a practical case, consider a deep-sea depth $H = 4000$ m: the (phase or group) velocity would be $\approx$200 m s$^{-1} \approx 713$ km h^{-1}. Thus, a tsunami in a deep-sea area can travel over very long distances within minutes or hours, with the speed of a jet plane. For the deepest ocean areas, the speed could exceed 1180 km h^{-1}.

When the tsunami approaches a coastal region, the speed changes as the square root of the depth. For example, if $H = 10$ m, then the speed becomes $\approx$10 m $s^{-1} \approx$ 36 km h^{-1}. Even with this drastic reduction, the wave speed is *largely sufficient to outrun a swimmer or a running person*.

Equation (16) has other important consequences. Specifically, for a constant frequency it implies that the wavelength $\lambda = 2\pi/k$ changes as $\sqrt{H}$. Consider a wave with $\lambda = 100$ km = 10^5 m in a deep-sea area with $H = 4000$ m: the time required for the wave to travel along a distance equal to λ (i.e., the period) is $\approx 10^5/(200) \approx 500$ s ≈ 8 min. When $H = 10$ m, λ decreases to 5×10^3 m—however, the speed decreases at the same rate so that the time to travel along λ remains the same. This implies that the typical time separating the different events of a tsunami is always minutes or even hours, much longer than the period (5–20 s) of wind-generated waves. This can cause tragic mistakes in judging the tsunami behaviour when it reaches a beach: an initial harmless 'drawdown' (a sudden drop of the water height at the shore) can attract people to the beach—and then is followed, after a relatively long time, by a devastating gigantic wave that cannot be outrun.

The change in speed with the depth has a fundamental implication on the height of the wave. This point must be discussed in detail since there is often a wrong perception that the increase in height as the tsunami reaches the coast is due to the conservation of the water mass. This is not true since the tsunami does not cause a translation of water but only a propagating perturbation of the *local* motion of water particles.

The increase in height as the tsunami reaches a shore is instead due to the conservation of energy [2]. With neither viscosity nor other losses, the energy flow associated with the wave must remain constant. Equations (11) and (14) show that the wave energy is proportional to the square of δ_{yo}—a parameter that in turn determines the wave height. Thus, the *energy flow* is proportional to $\delta_{yo}^2 v_g$ and therefore (equation (17)) to $\delta_{yo}^2\sqrt{H}$: a constant energy flow requires the wave height to be proportional to $H^{-1/4}$.

The consequences are striking. Consider a tall and devastating tsunami wave with $\delta_{yo} \approx 15$ m near the shore, say for $H \approx 2$ m. The corresponding height in a deep-sea area with $H \approx 4 \times 10^3$ m is smaller by a factor $[2/(4 \times 10^3)]^{1/4} \approx 0.025$, which gives $\delta_{yo} \approx 0.38$ m = 38 cm—a rather normal wave, very difficult to note. Thus, the terrible nature of the tsunami becomes visible only at the last minute and this complicates the early warning procedures. On the other hand, the moderate wave height far from the shore opens the way to an effective defensive strategy for boats (and even people): moving to deeper waters.

Finally, it should be stressed that the hypothesis of no energy losses is, of course, unrealistic: losses do occur due to a variety of causes, including internal friction and solid-water friction related to viscosity. Such losses are proportional to the speed. According to equations (7), (8), (12) and (13), the speed is proportional to the frequency and therefore to $1/\lambda$. The very long wavelength of tsunamis thus corresponds to limited losses: the waves can travel over very large distances and cause catastrophic damage and loss of life very far from their original cause.

In summary, our simple derivation of the dispersion law should enable the students to understand the physics background of the key features of tsunamis. In particular:

- The wavelength of tsunamis is so long that they always behave as shallow-water waves even in deep-sea areas.
- As for all shallow-water waves, their propagation speed is proportional to $\sqrt{gH}$, making them very fast in deep waters and still quite fast close to the shore.

- The typical time intervals between tsunami-related phenomena are of minutes to hours, so that warning signs (such as earth shaking from a strong faraway earthquake or the anomalous retreat of the sea water from the shore) leave sufficient time for life-saving actions such as going to high ground or at least leaving the beach. On the other hand, the long time delay can be deceiving and lead to tragic mistakes.
- The wave height can be gigantic close to the shore, but it is much smaller far at sea. This makes the tsunamis difficult to detect but it also allows defensive actions such as moving far from the coast and into the sea.

The presentation to the students could be completed by a rough calculation of the energy carried by the wave per unit time and unit surface, compared for example to that of a fast-moving truck.

3. Final remarks

The background for our extremely simple derivation is quite sound. The motion equations (5) and (6) are first-order linear expansion in terms of $ky \ll 1$ of the hyperbolic functions in the more advanced solutions of the wave equation. Therefore, the conceptual framework is equivalent and based [3] on the continuity equation and the irrotational character of the local motion (although this last hypothesis could be replaced, according to [1], by plausibility arguments on the liquid particle trajectories)[2].

We believe, however, that the mathematical simplifications make the treatment much easier to follow for the targeted audience, without resorting to the drastic and unsatisfactory tactics of introducing the dispersion law (and/or its consequences) with no justification and no physical background. The derivation reveals that the basic physics of these dreadful phenomena is rather simple. At the same time, its results show that the absence of simple life-saving measures is absolutely unjustifiable.

Acknowledgments

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References

[1] Behroozi F and Podolefsky N 2001 *Eur. J. Phys.* **23** 225

[2] See, for example, www.geophys.washington.edu/tsunami/general/physics/physics.html of the University of Washington; and www.wcatwc.gov/physics.htm of the West Coast & Alaska Tsunami Warning Center, NOAA (a US government agency).

[3] See, for example, Thorne K S *Applications of Classical Physics* chapter 15 (available in pdf form in the site www.pma.caltech.edu/Courses/ph136/yr2002/index.html of Caltech), or any medium-level textbook in hydrodynamics.

[2] Specifically, the velocity components in equations (7) and (8) correspond in the limit $ky \ll 1$ to the velocity potential $\psi = \psi_o \cosh(ky) \exp[\mathrm{i}(kx \pm \omega t)]$. This is the standard solution of the Laplace equation for water waves with the boundary condition of zero vertical velocity at the sea bottom.